3.5 Higher – Order Derivatives, Velocity and Acceleration

If the derivative is itself a continuous and differentiable function then it too can be differentiated and its derivative would be the second derivative of the original function.

Higher order derivatives are determined by simply repeating the process of differentiation.

Let $y = f(x)$. Then the second derivative is denoted by:

$$ y'' = \frac{dy'}{dx} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2y}{dx^2} \quad \text{(or } D^2y) \quad \text{or} $$

Here the rules of differentiation are simply applied to $y'$.

By extension: $y''' = \frac{d^3y}{dx^3}$.

In general, for the $n$th derivative: $y^{(n)} = \frac{d}{dx} y^{(n-1)}$.

The brackets in the superscript serve to distinguish between differentiation and a power.

Example 1: Let $y = 3x^5 - \frac{1}{x}$. Find $y''$ and $y'''$.

Solution:

$$ y' = 15x^4 - (-x^{-2}) = 15x^4 + 1/x^2; \quad y'' = 60x^3 - 2x^{-3} = 60x^3 - 2/x^3 $$

$$ y''' = 180x^2 + 6x^{-4} = 180x^2 + 6/x^4 $$

• instantaneous rates of change

In many important applications, a dependent variable $y$ has a non-linear relationship with an independent variable $x$. It is often necessary, in such cases, to determine the instantaneous rate of change of $y$ with respect to $x$.

If $y = f(x)$ then $\frac{dy}{dx}$ evaluated at $x = a$, is the instantaneous rate of change of $y$ with respect to $x$ at $a$. 
motion along a line

Let $s = s(t)$ be the position $s$ of an object moving along a line at time $t$.

1. The **displacement** of the object over the time interval $t$ to $t + \Delta t$ is
   $$\Delta s = s(t + \Delta t) - s(t).$$

2. The **average velocity** of the object over the same time interval is
   $$v_{av} = \frac{\Delta s}{\Delta t} = \frac{s(t + \Delta t) - s(t)}{\Delta t}.$$  
   
3. The **instantaneous velocity** at time $t$ is $v(t) = \frac{ds}{dt}$.

4. The **speed** of the object is: $\text{Speed} = |v(t)| = \left| \frac{ds}{dt} \right|$, the magnitude of the velocity.

5. The **acceleration** of the object at time $t$, $a(t)$, is its instantaneous rate of change of velocity at time $t$. That is, $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

Objects moving under the influence of gravity have positions functions of the form:
$$s(t) = \frac{a}{2} t^2 + v_0 t + s_0.$$  

Here $s_0 = s$ when $t = 0$: the initial position

$v_0 = v$ when $t = 0$: the initial velocity

$v(t) = \frac{ds}{dt} = at + v_0$.

$a = \text{the acceleration at time } t, a = \frac{d^2s}{dt^2}$.

For any general position function $s(t)$, $s$ is **increasing** when $v > 0$ and **decreasing** when $v < 0$.

An object in motion must have $v = 0$ at a time when it changes direction.

Similarly, $a < 0$ means $v$ is decreasing (the object is decelerating)

and $a > 0$ means $v$ is increasing (the object is accelerating).
Example 2: A particle starting from rest at time $t = 0$ moves with position function $s(t) = t^4 - t^2/2$.

(a) At what time does the particle change direction?

(b) What is the particle’s acceleration at the time found in part (a)?

solution: (a) To find times when the particle changes direction, first determine the times when the velocity $v = 0$.

$$s'(t) = v(t) = 4t^3 - t = t(4t^2 - 1) = t(2t + 1)(2t - 1) = 0$$

$\Rightarrow t = 0$ or $1/2$.

When $0 < t < 1/2$, $v < 0$, but when $t > 1/2$, $v > 0$.

The particle changes direction at $t = 1/2$.

(b) $a(t) = v'(t) = 12t^2 - 1$ so $a(1/2) = 2$.

The particle is accelerating in the positive direction when it changes direction.

$$v_{av} = \frac{s(6.49162) - s(1.49162)}{5} = \frac{0 - 190.5895}{5} = -38.1179$$
Example 3: A projectile is fired directly upward from the top ledge of a 102.9m high building with an initial velocity of 98m per second. The displacement, s in meters, of the projectile is given by \( s = s(t) = -4.9t^2 + 98t + 102.9 \).

(a) Find the velocity of the projectile at time \( t = a \).
(b) When is the velocity zero?
(c) How fast is the projectile moving when it hits the ground?
(d) What is the projectile’s average velocity over the 5 second interval immediately before it hits the ground?

**solution:**

(a) \( v(t) = s'(t) = -9.8t + 98 \)

(b) \( v(a) = -9.8a + 98 = 0 \) \( \Rightarrow \) \( a = 10 \)

The projectile stops climbing after 10 seconds.

(c) The projectile hits the ground when
\[
-4.9t^2 + 98t + 102.9 = 0
\]
\[
\Rightarrow -4.9(t^2 - 20t - 21) = -4.9(t - 21)(t + 1) = 0
\]
\[
\Rightarrow \text{so the projectile hits the ground after 21 seconds}
\]
\[
\Rightarrow \text{from (b) the velocity at the ground is } v(21) = -107.8 \text{m/s}
\]

(d) \[
\text{average velocity } = \frac{s(21) - s(16)}{5} = \frac{-416.5}{5} = -83.3 \text{m/s}
\]

The projectile takes only 1 second to travel the height of the building.
The roots of $s(t) = 0$ are $-1$ and $21$. The root of $-1$ is naturally discarded. However, it arises because the mathematics allows the process to go backward in time to a firing from the ground with an initial velocity of $107.8\text{m/s}$. Recall that this corresponds to the final velocity in magnitude but not in direction. Under ideal conditions (no air friction), the initial velocity at the ground has the same magnitude as the final velocity at the ground. In other words, the distance/time curve is a parabola.